

## ON THE TOPOLOGIES INDUCED BY A CONE

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ABSTRACT. Let  $A$  be a commutative and unital  $\mathbb{R}$ -algebra, and  $M$  be an Archimedean quadratic module of  $A$ . We define a submultiplicative seminorm  $\|\cdot\|_M$  on  $A$ , associated with  $M$ . We show that the closure of  $M$  with respect to  $\|\cdot\|_M$ -topology is equal to the closure of  $M$  with respect to the finest locally convex topology on  $A$ . We also compute the closure of any cone in  $\|\cdot\|_M$ -topology. Then we omit the Archimedean condition and show that there still exists a lmc topology associated to  $M$ , pursuing the same properties.

## 1. INTRODUCTION

The classical  $K$ -moment problem for a closed subset  $K$  of  $\mathbb{R}^n$ ,  $n \geq 1$ , is determining whether a given linear functional  $L : \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \dots, X_n] \rightarrow \mathbb{R}$  is representable as an integral with respect to a positive Radon measure  $\mu$ , supported on  $K$  or not. In symbols

$$L(f) = \int_K f \, d\mu.$$

An obvious necessary condition for existence of such a measure is that for every  $f \in \mathbb{R}[\underline{X}]$  with  $f \geq 0$  on  $K$ ,  $L(f)$  should be non-negative. In 1936, Haviland proved that this necessary condition is also sufficient [7, 8]:

**Theorem 1.1** (Haviland). *A linear function  $L : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$  is representable as an integral with respect to a positive Radon measure  $\mu$  on  $K$  if and only if  $L(\text{Psd}(K)) \subseteq \mathbb{R}_{\geq 0}$ .*

Here,  $\text{Psd}(K) = \{f \in \mathbb{R}[\underline{X}] : f(x) \geq 0 \ \forall x \in K\}$  and  $\mathbb{R}_{\geq 0} = [0, \infty)$ .

The major flaw of Haviland's result is that the structure of  $\text{Psd}(K)$  is usually very complicated and hence checking non-negativity of  $L$  on  $\text{Psd}(K)$  is practically infeasible.

Schmüdgen assumed  $K$  to be a basic compact semialgebraic sets and solved the  $K$ -moment problem in this particular case effectively: A set  $K \subseteq \mathbb{R}^n$  is called basic closed semialgebraic, if there exists a finite set of polynomials  $S = \{g_1, \dots, g_m\}$  such that

$$K = \mathcal{K}_S = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, \dots, m\}.$$

The preordering generated by  $S$ , denoted by  $T_S$ , is the subset of  $\mathbb{R}[\underline{X}]$ , consisting of all polynomials  $f \in \mathbb{R}[\underline{X}]$ , such that

$$f = \sigma_0 + \sum_{e \in \{0,1\}^m} \sigma_e \underline{g}^e,$$

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where  $\sigma_0, \sigma_e$ , for  $e = (e_1, \dots, e_m) \in \{0, 1\}^m$  are finite sums of squares of polynomials and  $\underline{g}^e := g_1^{e_1} \dots g_m^{e_m}$ .

**Theorem 1.2** (Schmüdgen [9]). *Let  $K = \mathcal{K}_S$  be a basic compact semialgebraic subset of  $\mathbb{R}^n$  and  $L : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$  a functional. If  $L(T_S) \subseteq \mathbb{R}_{\geq 0}$ , then  $L(\text{Psd}(K)) \subseteq \mathbb{R}_{\geq 0}$ .*

Now, since elements of  $T_S$  are finitely representable by polynomials in  $S$ , checking non-negativity of  $L$  over  $T_S$  is practical and if  $\mathcal{K}_S$  is compact, Schmüdgen's theorem guarantees non-negativity of  $L$  on  $\text{Psd}(\mathcal{K}_S)$ . Therefore, in this case, one can use Haviland's theorem to deduce existence of a representing Radon measure for  $L$ .

A closer look at theorem 1.2 reveals an equivalent topological statement. Let  $\varphi$  be the finest non-discrete locally convex topology on  $\mathbb{R}[\underline{X}]$ . Then 1.2 can be read as

$$(1) \quad \overline{T_S}^\varphi = \text{Psd}(\mathcal{K}_S).$$

We explain this in Remark 4.2.

Following the topological approach, Berg and Maserick in [2] showed that

$$(2) \quad \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \text{Psd}([-1, 1]^n),$$

where  $\sum \mathbb{R}[\underline{X}]^2$  is the set of all finite sums of squares of polynomials. In terms of moments, if  $L$  is positive semidefinite and  $\|\cdot\|_1$ -continuous, then  $L$  admits integral representation by a Radon measure on  $[-1, 1]^n$ , where  $\|\sum_\alpha f_\alpha \underline{X}^\alpha\|_1 = \sum_\alpha |f_\alpha|$ . They also generalized this for weighted  $\|\cdot\|_1$ -topologies. In both (1) and (2), the left side of the equality is the closure of a cone and the right side is  $\text{Psd}(K)$  for some  $K \subseteq \mathbb{R}^n$ . Relaxing the relation between objects of these equations, in [4] we started to study the following general equation:

$$(3) \quad \overline{C}^\tau = \text{Psd}(K),$$

where  $C \subseteq \mathbb{R}[\underline{X}]$  is a cone,  $\tau$  a locally convex topology on  $\mathbb{R}[\underline{X}]$  and  $K \subseteq \mathbb{R}^n$  a closed set. It is also explained that if (3) holds, then every  $\tau$ -continuous linear functional with  $L(C) \subseteq \mathbb{R}_{\geq 0}$  admits an integral representation with respect to a Radon measure on  $K$ . Furthermore, we replaced  $\mathbb{R}[\underline{X}]$  with a unital commutative  $\mathbb{R}$ -algebra and  $\mathbb{R}^n$  with  $\mathcal{X}(A)$ , the set of all real valued  $\mathbb{R}$ -algebra homomorphisms on  $A$ , equipped with subspace topology, where  $\mathcal{X}(A)$  is considered as a subspace of  $\mathbb{R}^A$ , with product topology.

In section 2, we briefly review the solutions of (3), studied in [3] and [4]. First we fix a closed set  $K \subseteq \mathcal{X}(A)$  and solve (3) for given cones  $C$ , in terms of the topology  $\tau$ , which slightly generalizes results of [3]. Then we fix a locally multiplicatively convex topology  $\tau$  on  $A$  and for a given cone  $C$  we solve (3) in terms of  $K$ .

In section 3, we associate a topology to any Archimedean cone  $C$  and study its properties. Then in section 4, we fix an Archimedean cone  $C$  and for given sets  $K$ , solve (3) in terms of the topology  $\tau$ , introduced in section 3. Moreover, we generalize our results to the case where  $C$  is not Archimedean.

## 2. SOLUTIONS OF (3) FOR A FIXED $K$ OR A FIXED TOPOLOGY

In this section we briefly review known solutions of (3), studied in [3] and [4]. We begin by introducing terms and notations that will be used in this article.

From now on, we always assume that  $A$  is a unital commutative  $\mathbb{R}$ -algebra. A cone of  $A$  is a set  $C \subseteq A$  such that

$$C + C \subseteq C, \quad \mathbb{R}_{\geq 0} \cdot C \subseteq C.$$

A quadratic module  $M$ , is a cone, containing 0 and 1 which is closed under multiplication by sums of squares; i.e.,

$$\sum A^2 \cdot M \subseteq M.$$

A cone  $C$  is said to be *Archimedean*, if for every  $a \in A$ , there exists  $r \in \mathbb{R}_{\geq 0}$  such that  $r \pm a \in C$ . If a quadratic module  $M$ , is also closed under multiplication (i.e.  $M \cdot M \subseteq M$ ) then we say that  $M$  is a preordering.

Suppose that  $M$  is a quadratic module, then it is easy to see that  $I = M \cap -M$  is an ideal of  $A$ : Clearly  $0 \in I$  and  $a^2 I \subseteq I$  for every  $a \in A$ . Thus

$$aI = \left( \left( \frac{a+1}{2} \right)^2 - \left( \frac{a-1}{2} \right)^2 \right) I \in I - I \subseteq I.$$

The ideal  $I$  is called the *support* of  $M$ . Clearly,  $M$  is a proper subset of  $A$  if and only if  $-1 \notin M$ .

The set of all real valued  $\mathbb{R}$ -algebra homomorphisms on  $A$  is denoted by  $\mathcal{X}(A)$ . If  $\tau$  is a locally convex topology on  $A$ , the set of all  $\tau$ -continuous elements of  $\mathcal{X}(A)$  will be denoted by  $\mathfrak{sp}_\tau(A)$  which is known in the literature as the Gelfand spectrum of  $(A, \tau)$ . Every element  $a$  of  $A$  induces a map on  $\mathcal{X}(A)$ , in the following way:

$$\begin{aligned} \hat{a} : \mathcal{X}(A) &\longrightarrow \mathbb{R} \\ \alpha &\mapsto \alpha(a). \end{aligned}$$

We denote the set of all elements of  $\mathcal{X}(A)$  that are non-negative on  $M$  by  $\mathcal{K}_M$ . In symbols:

$$\mathcal{K}_M := \{ \alpha \in \mathcal{X}(A) : \alpha(M) \subseteq \mathbb{R}_{\geq 0} \}.$$

**Definition 2.1.**

- (1) A set  $U \subseteq A$  is multiplicatively closed, if  $U \cdot U \subseteq U$ .
- (2) A *locally multiplicatively convex topology* (lmc) on  $A$  is a locally convex topology which admits a system of neighbourhoods at 0, consisting of multiplicatively closed convex sets.
- (3) A seminorm  $\rho$  on  $A$  is called *submultiplicative*, if  $\rho(ab) \leq \rho(a)\rho(b)$ , for all  $a, b \in A$ .

We recall the following result about lmc topologies.

**Theorem 2.2.** *A locally convex vector space topology  $\tau$  on  $A$  is lmc if and only if  $\tau$  is generated by a family of submultiplicative seminorms on  $A$ .*

*Proof.* See [1, §4.3-2]. □

The following result from [4] will be used in what follows:

**Theorem 2.3.** *Let  $\rho$  be a submultiplicative seminorm on  $A$  and  $M$  be a quadratic module of  $A$ . Then  $\overline{M}^\rho = \text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}_\rho(A))$ .*

*Proof.* See [4, Theorem 3.7]. □

**2.1. The case where  $K$  is fixed.** Now let's fix a set  $\emptyset \neq X \subseteq \mathcal{X}(A)$ . We introduce a topology  $\mathcal{T}_X$  on  $A$  such that (3) holds for  $C$  to be any quadratic module  $M$ ,  $\tau = \mathcal{T}_X$  and  $K = X \cap \mathcal{K}_M$ ; i.e.,

$$\overline{M}^{\mathcal{T}_X} = \text{Psd}(X \cap \mathcal{K}_M).$$

Since  $X \neq \emptyset$ , the family  $k(X) = \{D \subseteq X : \emptyset \neq D \text{ is compact}\}$  is non-empty. To every  $D \in k(X)$  we assign a submultiplicative seminorm  $\rho_D$  defined by

$$\rho_D(a) = \sup_{\alpha \in D} |\hat{a}(\alpha)|.$$

**Definition 2.4.** The family  $\{\rho_D\}_{D \in k(X)}$  induces a locally multiplicatively convex topology  $\mathcal{T}_X$ , on  $A$ .

We note that the topology  $\mathcal{T}_X$  defined above is slightly different from the one defined in [3]. In the former, the topology  $\mathcal{T}_X$  is induced by the family of seminorms, induced by evaluations on each single point of  $X$ , i.e.,  $k(X) = \{\{\alpha\} : \alpha \in X\}$ .

**Lemma 2.5.** Let  $D \subseteq \mathcal{X}(A)$  be a compact set. Then  $\mathfrak{sp}_{\rho_D}(A) = D$ .

*Proof.* Note that an element  $\alpha \in \mathcal{X}(A)$  is  $\rho_D$ -continuous if and only if  $|\alpha(a)| \leq \rho_D(a)$  for all  $a \in A$ . Thus every  $\alpha \in D$  is  $\rho_D$ -continuous. Let  $\beta \in \mathcal{X}(A) \setminus D$  and  $D' = D \cup \{\beta\}$ . Since  $\mathcal{X}(A)$  is completely regular, there exists a continuous function  $f \in C(\mathcal{X}(A))$  such that  $f(\beta) = 1$  and  $f|_D = 0$ . Note that  $\rho_D$  is also extendible to  $C(\mathcal{X}(A))$  naturally by defining  $\rho_D(g) = \sup_{\alpha \in D} |g(\alpha)|$  for each  $g \in C(\mathcal{X}(A))$ . By Stone–Weierstrass theorem, for every  $\epsilon > 0$ , there exist  $a_\epsilon \in A$  such that  $\rho_{D'}(f - a_\epsilon) < \epsilon$ . Clearly  $|1 - \beta(a_\epsilon)| \leq \epsilon$  and  $\rho_D(a_\epsilon) \leq \epsilon$ . So for  $\epsilon < \frac{1}{2}$ , we get  $\rho_D(a_\epsilon) < |\beta(a_\epsilon)|$  which violates the necessary and sufficient condition for continuity of  $\beta$ . Thus  $\beta \notin \mathfrak{sp}_{\rho_D}(A)$ .  $\square$

**Corollary 2.6.**  $\mathfrak{sp}_{\mathcal{T}_X}(A) = X$ .

*Proof.*

$$\begin{aligned} \mathfrak{sp}_{\mathcal{T}_X}(A) &= \bigcup_{D \in k(X)} \mathfrak{sp}_{\rho_D}(A) \\ &= \bigcup_{D \in k(X)} D \\ &= X. \end{aligned}$$

$\square$

**Theorem 2.7.** Let  $M$  be a quadratic module of  $A$  and  $X \subseteq \mathcal{X}(A)$ . Then  $\overline{M}^{\mathcal{T}_X} = \text{Psd}(X \cap \mathcal{K}_M)$ .

*Proof.* Applying theorem 2.3, for every  $D \in k(X)$ ,  $\overline{M}^{\rho_D} = \text{Psd}(D \cap \mathcal{K}_M)$ . Therefore

$$\begin{aligned} \overline{M}^{\mathcal{T}_X} &= \bigcap_{D \in k(X)} \overline{M}^{\rho_D} \\ &= \bigcap_{D \in k(X)} \text{Psd}(D \cap \mathcal{K}_M) \\ &= \text{Psd}\left(\bigcup_{D \in k(X)} D \cap \mathcal{K}_M\right) \\ &= \text{Psd}(X \cap \mathcal{K}_M), \end{aligned}$$

as desired.  $\square$

**2.2. The case where  $\tau$  is fixed.** We now review the situation where a lmc topology  $\tau$  on  $A$  is fixed, a quadratic module  $M$  is given, and solve (3) for  $K$  as it is explained in [4, §5].

Suppose that  $\tau$  is a lmc topology. By theorem 2.2, there exists a family  $\mathcal{F}$  of submultiplicative seminorms, inducing  $\tau$  on  $A$ . For  $\rho_1, \rho_2 \in \mathcal{F}$ , the map defined by  $\rho(a) = \max\{\rho_1(a), \rho_2(a)\}$  is again a submultiplicative seminorm and the topology induced by  $\mathcal{F} \cup \{\rho\}$  is again equal to  $\tau$ . Inductively, if we add the maximum of any finite number of elements of  $\mathcal{F}$  to it, the resulting topology will not change. A family of seminorms which contains the maximum of all finite sets of its elements is called saturated. Clearly every family of seminorms can be completed to a saturated one. The advantage of working with saturated families over non saturated is explained in the next proposition.

**Proposition 2.8.** *Suppose  $\tau$  is an lmc topology on  $A$  generated by a saturated family  $\mathcal{F}$  of submultiplicative seminorms of  $A$ . Then  $\mathfrak{sp}_\tau(A) = \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}_\rho(A)$ .*

*Proof.* [1, §4.10-7]. □

Implementing the same argument we used in theorem 2.7 for a saturated family of submultiplicative seminorms  $\mathcal{F}$  inducing  $\tau$  and a quadratic module  $M$ , we have,

$$\begin{aligned} \overline{M}^\tau &= \bigcap_{\rho \in \mathcal{F}} \overline{M}^\rho \\ &= \bigcap_{\rho \in \mathcal{F}} \text{Psd}(\mathfrak{sp}_\rho(A) \cap \mathcal{K}_M) \\ &= \text{Psd}(\bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}_\rho(A) \cap \mathcal{K}_M) \\ &= \text{Psd}(\mathfrak{sp}_\tau(A) \cap \mathcal{K}_M), \end{aligned}$$

which proves the following:

**Theorem 2.9.** *Let  $\tau$  be an lmc topology on  $A$  and let  $M$  be any quadratic module of  $A$ . Then  $\overline{M}^\tau = \text{Psd}(\mathfrak{sp}_\tau(A) \cap \mathcal{K}_M)$ .*

### 3. THE SEMINORM INDUCED BY A CONE

Fixing an Archimedean quadratic module  $M$  of  $A$ , we associate a non-negative function  $\|\cdot\|_M$  to  $M$ , defined on  $A$ . We prove that it is in fact a submultiplicative seminorm on  $A$  and study some of its basic properties, such as its relation to the finest locally convex topology on  $A$  in the following sections.

**Definition 3.1.** Let  $M$  be an Archimedean quadratic module of  $A$ . For every  $a \in A$ , define  $\|a\|_M$  by

$$\|a\|_M := \inf\{r \in \mathbb{R} : r \pm a \in M\}.$$

We will make use of the following well-known result of T. Jacobi to prove some basic properties of  $\|\cdot\|_M$ :

**Theorem 3.2** (Jacobi). *Suppose  $M$  is an Archimedean quadratic module of  $A$ . Then for each  $a \in A$ ,*

$$\hat{a} > 0 \text{ on } \mathcal{K}_M \Rightarrow a \in M.$$

*Proof.* See [6, Theorem 4]. □

**Proposition 3.3.** *For all  $a \in A$ ,  $\|a\|_M \geq 0$ .*

*Proof.* There are two possible cases:

*Case 1:*  $a \in M$  or  $-a \in M$ . Without loss of generality, assume that  $a \in M$ . Then for all  $\alpha \in \mathcal{K}_M$ ,  $\hat{a}(\alpha) \geq 0$ . Therefore, for any real number  $r > 0$ ,  $\widehat{r+a} > 0$  on  $\mathcal{K}_M$  and hence by theorem 3.2,  $r+a \in M$ .

If also  $-a \in M$ , then  $\alpha(a) = 0$  for all  $\alpha \in \mathcal{K}_M$ . Thus if  $r < 0$ , then  $r \pm a \notin M$  and  $\|a\|_M = 0$ .

Suppose that  $-a \notin M$ . Then, there exists  $\alpha \in \mathcal{K}_M$  such that  $\alpha(-a) \leq 0$ . Therefore  $r-a \notin M$  for any  $r \leq 0$  and hence  $\|a\|_M > 0$ .

*Case 2:*  $\pm a \notin M$ . By theorem 3.2, there exist  $\alpha, \beta \in \mathcal{K}_M$  such that  $\alpha(a) \leq 0$  and  $\beta(-a) \leq 0$ . Therefore,  $\alpha(r+a), \beta(r-a) \leq 0$  for every  $r \leq 0$  which implies that  $r \pm a \notin M$ . So the set  $\{r \in \mathbb{R} : r \pm a \in M\}$  is bounded below by 0. Hence  $\|a\|_M$  exists and is non-negative.  $\square$

In fact  $\|\cdot\|_M$  induces a submultiplicative seminorm on  $A$  (A seminorm  $\rho$  on  $A$  is said to be submultiplicative if  $\rho(a \cdot b) \leq \rho(a)\rho(b)$  for all  $a, b \in A$ ).

**Proposition 3.4.**  $\|\cdot\|_M$  is a submultiplicative seminorm on  $A$ .

*Proof.* By proposition 3.3,  $\|0\|_M = 0$  and the range of  $\|\cdot\|_M$  consists of non-negative real numbers.

$$(1) \quad \forall \lambda \in \mathbb{R} \quad \forall a \in A \quad \|\lambda a\|_M = |\lambda| \|a\|_M:$$

Clearly for  $\lambda = 0$ ,  $\|\lambda a\|_M = \|0\|_M = 0 \times \|a\|_M = 0$ . Suppose that  $\lambda \neq 0$ .

Then

$$\begin{aligned} \{r : r \pm \lambda a \in M\} &= \{r : |\lambda|(\frac{r}{|\lambda|} \pm a) \in M\} \\ &= \{r : \frac{r}{|\lambda|} \pm a \in \frac{1}{|\lambda|}M\} \\ &= |\lambda|\{r : r \pm a \in M\}. \end{aligned}$$

So,  $\|\lambda a\|_M = |\lambda| \|a\|_M$ .

$$(2) \quad \forall a, b \in A \quad \|a+b\|_M \leq \|a\|_M + \|b\|_M:$$

For any  $\epsilon > 0$  and  $a, b \in A$ , we have

$$\|a\|_M + \|b\|_M + \epsilon \in \{r : r \pm (a+b) \in M\}.$$

Hence,  $\sup\{r : r \pm (a+b) \in M\} \leq \|a\|_M + \|b\|_M + \epsilon$ , for every  $\epsilon > 0$ . So,  $\|a+b\|_M \leq \|a\|_M + \|b\|_M$ .

$$(3) \quad \forall a, b \in A \quad \|ab\|_M \leq \|a\|_M \|b\|_M:$$

For any  $\epsilon > 0$ ,  $(\|a\|_M + \epsilon) \pm a, (\|b\|_M + \epsilon) \pm b \in M$ . Therefore  $|\alpha(a)| < \|a\|_M + \epsilon$  and  $|\alpha(b)| < \|b\|_M + \epsilon$  for all  $\alpha \in \mathcal{K}_M$ . So

$$|\alpha(ab)| < (\|a\|_M + \epsilon)(\|b\|_M + \epsilon),$$

and hence  $(\|a\|_M + \epsilon)(\|b\|_M + \epsilon) \pm ab \in M$  by 3.2. Therefore  $\|ab\|_M \leq (\|a\|_M + \epsilon)(\|b\|_M + \epsilon)$  for any  $\epsilon > 0$ . Letting  $\epsilon \rightarrow 0$ , we get

$$\|ab\|_M \leq \|a\|_M \|b\|_M,$$

as desired.  $\square$

Proposition 3.4, simply asserts that if  $M \subseteq A$  is an Archimedean quadratic module then the pair  $(A, \|\cdot\|_M)$  is a seminormed  $\mathbb{R}$ -algebra.

## 4. CLOSURE WITH RESPECT TO THE INDUCED SEMINORM

In this section we study the relation between  $\|\cdot\|_M$ -continuity and positivity of linear functionals on  $M$ , when  $M$  is an Archimedean quadratic module. Then we find the  $\|\cdot\|_M$ -closure of  $M$ . This gives a solution for integral representability of positive semidefinite  $\|\cdot\|_M$ -continuous functionals.

**Theorem 4.1.** *Let  $M$  be an Archimedean quadratic module. If a linear functional  $L : A \rightarrow \mathbb{R}$  is non-negative on  $M$ , then  $L$  is  $\|\cdot\|_M$ -continuous.*

*Proof.* Since  $M$  is Archimedean, for every  $a \in A$ , there exists  $r \geq 0$  such that  $r \pm a \in M$ . By positivity of  $L$  on  $M$ , we have  $L(r \pm a) \geq 0$ . So,  $|L(a)| \leq L(r) = rL(1)$  which by definition means  $\|a\|_M \leq r$ . Therefore, for every  $a \in A$  we have  $|L(a)| \leq L(1) \cdot \|a\|_M$  and so,  $L$  is  $\|\cdot\|_M$ -continuous.  $\square$

**Remark 4.2.** Let  $\tau$  be a locally convex topology on  $A$  and  $C \subseteq A$  be a cone in  $A$ . Let

$$C_\tau^\vee := \{L : A \rightarrow \mathbb{R} : L \text{ is } \tau\text{-continuous and } L(C) \subseteq \mathbb{R}_{\geq 0}\},$$

and

$$C_\tau^{\vee\vee} := \{a \in A : L(a) \geq 0 \quad \forall L \in C_\tau^\vee\}.$$

One can show that  $C_\tau^{\vee\vee} = \overline{C}^\tau$ :

If  $b \notin \overline{C}^\tau$ , then there exists a convex open set  $O$  containing  $b$  and disjoint from  $C$ . By Banach separation theorem, there exists  $L \in C_\tau^\vee$  such that  $L < 0$  on  $O$ . Hence  $b \notin C_\tau^{\vee\vee}$  which proves  $C_\tau^{\vee\vee} \subseteq \overline{C}^\tau$ . The reverse inclusion is clear. Note that  $C \subseteq C_\tau^{\vee\vee}$ , therefore  $\overline{C}^\tau \subseteq \overline{C_\tau^{\vee\vee}}^\tau$  and

$$C_\tau^{\vee\vee} = \bigcap_{L \in C_\tau^\vee} L^{-1}(\mathbb{R}_{\geq 0}),$$

is  $\tau$ -closed; i.e.,  $\overline{C_\tau^{\vee\vee}}^\tau = C_\tau^{\vee\vee}$ .

**Corollary 4.3.** *Let  $M$  be an Archimedean quadratic module and  $\varphi$  be the finest locally convex topology on  $A$ . Then  $\overline{M}^\varphi = \overline{M}^{\|\cdot\|_M}$ .*

*Proof.* Since  $\varphi$  is the finest locally convex topology on  $A$ , every linear functional on  $A$  is  $\varphi$ -continuous. By definition  $M_\varphi^\vee = \{L : A \rightarrow \mathbb{R} : L(M) \subseteq \mathbb{R}_{\geq 0}\}$ . Since  $M$  is Archimedean, by theorem 4.1,  $L(M) \subseteq \mathbb{R}_{\geq 0}$  implies that  $L$  is  $\|\cdot\|_M$ -continuous, so  $M_\varphi^\vee = M_{\|\cdot\|_M}^\vee$ . Therefore,  $M_\varphi^{\vee\vee} = M_{\|\cdot\|_M}^{\vee\vee}$  and by applying Remark 4.2, we get  $\overline{M}^\varphi = \overline{M}^{\|\cdot\|_M}$ .  $\square$

**Theorem 4.4.** *Let  $M$  be an Archimedean quadratic module of  $A$  and  $T$  a cone such that  $\mathcal{K}_M \cap \mathcal{K}_T \neq \emptyset$ . Then*

$$\overline{T}^{\|\cdot\|_M} = \text{Psd}(\mathcal{K}_M \cap \mathcal{K}_T).$$

*Proof.* ( $\subseteq$ ) Clearly  $T \subseteq \text{Psd}(\mathcal{K}_T) \subseteq \text{Psd}(\mathcal{K}_M \cap \mathcal{K}_T)$ . Since every  $\alpha \in \mathcal{K}_T \cap \mathcal{K}_M$  is  $\|\cdot\|_M$ -continuous and

$$\text{Psd}(\mathcal{K}_M \cap \mathcal{K}_T) = \bigcap_{\alpha \in \mathcal{K}_M \cap \mathcal{K}_T} \alpha^{-1}(\mathbb{R}_{\geq 0}),$$

we see that  $\text{Psd}(\mathcal{K}_M \cap \mathcal{K}_T)$  is  $\|\cdot\|_M$ -closed. Therefore  $\overline{T}^{\|\cdot\|_M} \subseteq \text{Psd}(\mathcal{K}_M \cap \mathcal{K}_T)$ .

( $\supseteq$ ) Let  $A_1^* = \{L \in A^* : L(1) = 1\}$ , where  $A^*$  is the dual of  $(A, \|\cdot\|_M)$  equipped with weak-\* topology. The set  $T_{\|\cdot\|_M, 1}^\vee := A_1^* \cap T_{\|\cdot\|_M}^\vee$  is a convex closed subset of

a bounded closed ball of  $A^*$ , which is compact by Banach-Alaoglu theorem. By Krein-Milman theorem,  $T_{\|\cdot\|_M,1}^\vee$  is the weak-\* closure of convex hull of its extreme points.

**Claim.** If  $L$  is an extreme point of  $T_{\|\cdot\|_M,1}^\vee$ , then  $L \in \mathcal{K}_T \cap \mathcal{K}_M$ .

*Proof of the Claim:* For any  $c \in A$  such that  $L(c) \neq 0$  the map defined by  $L_c(x) = \frac{L(cx)}{L(c)}$  is a  $\|\cdot\|_M$ -continuous linear functional and  $L_c(1) = 1$ . If  $L \in T_{\|\cdot\|_M,1}^\vee$  and  $L(a), L(b) > 0$ , then

$$(4) \quad L(a) \cdot L_a(x) + L(b) \cdot L_b(x) = L(a+b) \cdot L_{a+b}(x).$$

Therefore  $L_{a+b} = (\frac{L(a)}{L(a+b)})L_a + (\frac{L(b)}{L(a+b)})L_b$  is a convex combination of  $L_a$  and  $L_b$ . Since  $L$  is continuous, there exists  $C > 0$  such that for all  $x \in A$ ,  $|L(x)| \leq C\|x\|_M$ . Therefore, for some  $N > 0$ ,  $L(N+x), L(N-x) > 0$ . Rewriting (4) for  $N \pm x$ , yields:

$$L = L_{2N} = \lambda L_{N+x} + (1-\lambda)L_{N-x}.$$

By assumption,  $L$  is an extreme point, thus  $L = L_{N+x}$ . So for every  $y \in A$ ,

$$L(y) = L_{N+x}(y) = \frac{L(N \cdot y) + L(xy)}{L(N+x)} = \frac{N \cdot L(y) + L(xy)}{N + L(x)}.$$

Hence  $N \cdot L(y) + L(x)L(y) = N \cdot L(y) + L(xy)$  or  $L(xy) = L(x)L(y)$ , as claimed.

Denoting by  $\text{cov}(\mathcal{K}_T \cap \mathcal{K}_M)$ , the convex hull of  $\mathcal{K}_T \cap \mathcal{K}_M$  in  $T_{\|\cdot\|_M,1}^\vee$ , the claim simply states that the weak-\* closure of  $\text{cov}(\mathcal{K}_T \cap \mathcal{K}_M)$  is equal to  $T_{\|\cdot\|_M,1}^\vee$ .

Now to show the reverse inclusion, take  $a \notin \overline{T}^{\|\cdot\|_M}$ . Then, there exists  $L \in T_{\|\cdot\|_M}^\vee$  such  $L(a) < 0$ . Replacing  $L$  with  $\frac{1}{L(1)}L$  if necessary, we can assume that  $L \in T_{\|\cdot\|_M,1}^\vee$ . Note that every element  $a \in A$  defines a continuous functional  $\hat{a} : A^* \rightarrow \mathbb{R}$  which  $L \mapsto L(a)$ . If  $b \in \text{Psd}(\mathcal{K}_T \cap \mathcal{K}_M)$ , then  $\hat{b} \geq 0$  on  $\mathcal{K}_T \cap \mathcal{K}_M$ . By continuity,  $\hat{b} \geq 0$  on  $\overline{\text{cov}(\mathcal{K}_T \cap \mathcal{K}_M)}^* = T_{\|\cdot\|_M,1}^\vee$ . Therefore

$$a \notin \overline{T}^{\|\cdot\|_M} \Rightarrow a \notin \text{Psd}(\mathcal{K}_T \cap \mathcal{K}_M),$$

or equivalently,  $\text{Psd}(\mathcal{K}_T \cap \mathcal{K}_M) \subseteq \overline{T}^{\|\cdot\|_M}$ . This completes the proof.  $\square$

**Remark 4.5.** In the theorem 4.4, if we assume that  $T$  is a quadratic module which is a richer structure, then the proof could be simplified by directly applying theorem 2.3. We only need to note that  $\mathbf{sp}_{\|\cdot\|_M}(A) = \mathcal{K}_M$  which is proved implicitly in corollary 4.3.

## 5. NON-ARCHIMEDEAN CONES

In this section we consider the case where the quadratic module  $M$  is not Archimedean, but  $\mathcal{K}_M \neq \emptyset$ . The purpose of this section is to define a topology  $\mathcal{T}_M$  on  $A$  such that (3) holds.

**Lemma 5.1.** Let  $M_1, M_2 \subseteq A$  be Archimedean quadratic modules. Then

- (1)  $M_1 \cap M_2$  is Archimedean;
- (2) If  $M_1 \subseteq M_2$  then the identity map  $\iota : (A, \|\cdot\|_{M_1}) \rightarrow (A, \|\cdot\|_{M_2})$  is continuous.



*Proof.* (1) Since  $M_1$  and  $M_2$  are Archimedean, for every  $a \in A$  there exist  $r_1, r_2 \in \mathbb{R}$  such that  $r_1 \pm a \in M_1$  and  $r_2 \pm a \in M_2$ . Take  $r = \max(r_1, r_2)$ , we have  $r \pm a \in M_1 \cap M_2$ .

(2) If  $M_1 \subseteq M_2$  then for all  $a \in A$  we have  $\|a\|_{M_2} \leq \|a\|_{M_1}$ . Thus the identity map  $\iota : (A, \|\cdot\|_{M_1}) \rightarrow (A, \|\cdot\|_{M_2})$  is continuous.  $\square$

Suppose that  $\mathcal{K}_M \neq \emptyset$ , then there always exists an Archimedean quadratic module  $M'$  containing  $M$ ; take  $\alpha \in \mathcal{K}_M$  and let

$$M' := \{a \in A : \hat{a}(\alpha) \geq 0\}.$$

Clearly  $M'$  is Archimedean (proof:  $|\hat{a}(\alpha)| \pm a \geq 0$  on  $\{\alpha\} = \mathcal{K}_{M'}$ ) and  $M \subseteq M'$ . Let

$$\text{Arch}(M) := \{T : T \text{ is Archimedean quadratic module and } M \subseteq T\},$$

then we can prove the following:

**Lemma 5.2.** *The family  $\{(A, \|\cdot\|_T) : T \in \text{Arch}(M)\}$  together with identity maps forms a direct system of seminormed algebras.*

*Proof.* The set  $\text{Arch}(M)$  is partially ordered by inclusion. Take  $T_1, T_2 \in \text{Arch}(M)$ , the quadratic module  $T_3 = T_1 \cap T_2$  contained in both  $T_1$  and  $T_2$  and also belongs to  $\text{Arch}(M)$ . The inclusion maps  $\iota_1 : (A, \|\cdot\|_{T_3}) \rightarrow (A, \|\cdot\|_{T_1})$  and  $\iota_2 : (A, \|\cdot\|_{T_3}) \rightarrow (A, \|\cdot\|_{T_2})$  are continuous by lemma 5.1. So  $\{(A, \|\cdot\|_T) : T \in \text{Arch}(M)\}$  together with inclusion maps is a direct system.  $\square$

The weakest topology on  $A$  such that all maps  $\iota : A \rightarrow (A, \|\cdot\|_T)$ ,  $T \in \text{Arch}(M)$  are continuous, coincides with the direct limit topology of  $\{(A, \|\cdot\|_T) : T \in \text{Arch}(M)\}$  on  $A$ . We denote this topology with  $\mathcal{T}_M$ . In symbols:

$$(A, \mathcal{T}_M) = \varinjlim_{T \in \text{Arch}(M)} (A, \|\cdot\|_T).$$

**Theorem 5.3.** *Let  $M$  be a quadratic module and  $C$  a cone in  $A$  such that  $\mathcal{K}_M \cap \mathcal{K}_C \neq \emptyset$ . Then  $\overline{C}^{\mathcal{T}_M} = \text{Psd}(\mathcal{K}_M \cap \mathcal{K}_C)$ .*

*Proof.* Since  $\mathcal{T}_M = \varinjlim_{T \in \text{Arch}(M)} (A, \|\cdot\|_T)$ , applying theorem 4.4 we have

$$\begin{aligned} \overline{C}^{\mathcal{T}_M} &= \bigcap_{T \in \text{Arch}(M)} \overline{C}^{\|\cdot\|_T} \\ &= \bigcap_{T \in \text{Arch}(M)} \text{Psd}(\mathcal{K}_T \cap \mathcal{K}_C) \\ &= \text{Psd}(\bigcup_{T \in \text{Arch}(M)} \mathcal{K}_T \cap \mathcal{K}_C) \\ &= \text{Psd}(\mathcal{K}_M \cap \mathcal{K}_C). \end{aligned}$$

$\square$

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